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Discontinuous Systems and the Henstock–Kurzweil Integral*

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By using the Henstock–Kurzweil integral and the inequalities of this integral, the existence and uniqueness theorems for the solution of the discontinuous Caratheodory system are established. The results are generalizations of earlier investigations [J. He and Po Chen, *Adv. in Math.* **16** (1987), 17–32 (in Chinese); A. F. Filippov, *Math. USSR-Sb.* **51** (1960) (in Russian); O. Hajek, *J. Differential Equations* **32** (1979), 149–185] of this discontinuous system.

1. INTRODUCTION

To introduce the problem which is considered in this paper, let us first consider the general discontinuous system defined by the following ordinary differential equation,

$$x' = f(t, x), \tag{1.1}$$

where $x = (x_1, x_2, \dots, x_n)^T$, $x' = dx/dt$, $f\colon G \to R^n$ is a function with some discontinuity, G is an open region in R^{n+1}.

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Because the first work on this general discontinuous system which was carried out by Caratheodory in 1918 [12], many investigators have studied this system. Furthermore, this discontinuous system has found many applications such as in the development of discontinuous oscillatory theory and in the theory of modern control. He and Chen [1] summarized some of the developments and their applications.

In the literature [1–3], the existence, uniqueness, and stability for the solution of the discontinuous Caratheodory and Filippov systems were obtained by using the Lebesgue integral and the solutions obtained are absolute continuous functions. However, there are discontinuous systems in which the right-hand side functions $f(t, x)$ are not Lebesgue integrable on certain intervals and their solutions are not absolute continuous functions. To illustrate, consider the following example:

EXAMPLE 1. Consider the following discontinuous system,

$$x' = t^2 x + h(t), \tag{1.2}$$

where $|t| \leq 1$, $|x| \leq 1$, and $h(t) = (d/dt)(t^2 \sin t^{-2})$. If $t \neq 0$ and $h(0) = 0$, then $f(t, x) = t^2 x(t) + h(t)$ is a highly oscillating function and is not Lebesgue integrable on $|t| \leq 1$. However, with $x(0) = 0$, the system (1.2) has the following solution,

$$x(t) = e^{t^3/3} \int_0^t e^{-(s^3/3)} h(s)\, ds. \tag{1.3}$$

The preceding integral is neither the Riemann integral nor the Lebesgue integral. It is the Henstock–Kurzweil integral and the solution $x(t)$ of system (1.2) is not an absolutely continuous function on $[0, 1]$.

In order to investigate this problem, we must use the Henstock–Kurzweil integral which encompasses the Newton, Riemann, and Lebesgue integrals [4–8]. This integral was introduced by Henstock and Kurzweil independently during 1957–1958 and was proven to be useful in the study of ordinary differential equations [4]. In this paper, the existence and uniqueness theorems for solutions of a generalized discontinuous Caratheodory system are established by using the Henstock–Kurzweil integral and the inequalities of this integral.

2. HENSTOCK–KURZWEIL INTEGRAL AND THE CONVERGENCE THEOREM

DEFINITION 2.1 [4–8]. Let $f: [a, b] \to R^n$ be a function. f is said to be Henstock–Kurzweil integrable to A on $[a, b]$ if for every $\epsilon > 0$, there exists a positive function $\delta(\xi)$ such that whenever a division D given by

$$a = t_0 < t_1 < \cdots < t_n = b \quad \text{and} \quad \{\xi_1, \xi_2, \dots, \xi_n\}$$

satisfies $\xi_i - \delta(\xi_i) < t_{i-1} \le \xi_i \le t_i < \xi + \delta(\xi_i)$ for $i = 1, 2, \ldots, k$, we have

$$\left\| \sum_{i=1}^{k} f(\xi_i)(t_i - t_{i-1}) - A \right\| < \epsilon, \tag{2.1}$$

where $\int_a^b f(t)\,dt = A$.

The Henstock–Kurzweil integral has all the standard properties one normally expects of any integral [4–8]. Here we only mention the relation between the Henstock–Kurzweil integral and the Lebesgue integral.

THEOREM 2.2 [4, 8]. *If f is Lebesgue integrable on $[a, b]$, then f is Henstock–Kurzweil integrable.*

THEOREM 2.3 [4, 8]. *If f is Henstock–Kurzweil integrable on $[a, b]$ and nonnegative, then f is Lebesgue integrable.*

DEFINITION 2.4 [8]. A function $F\colon [a, b] \to R^n$ is said to be absolutely continuous in the restricted sense on set X or, in short, $AC_*(X)$, if for every $\epsilon > 0$ there exists $\eta > 0$ such that for every finite or infinite sequence of nonoverlapping intervals $\{[a_i, b_i]\}$ with $a_i,\ b_i \in X$ and satisfying $\sum_i |b_i - a_i| < \eta$ we have $\sum_i \omega(F; [a_i, b_i]) < \epsilon$, where ω denotes the oscillation of F over $[a_i, b_i]$.

A function F is said to be generalized absolutely continuous in the restricted sense on $[a, b]$ or, ACG_*, if $[a, b]$ is the union of a sequence of closed sets X_i such that on each X_i, the function F is $AC_*(X_i)$.

Note that if F is $AC_*([a, b])$, then F is absolutely continuous on $[a, b]$. It is known that if F is differentiable everywhere on $[a, b]$ then F is ACG_* on $[a, b]$, see [14, Chap. VII, Theorem 10.5]. Hence, if there is a solution of $x' = f(t, x)$, for every t on $[a, b]$, then the solution is ACG_*.

EXAMPLE 2. Let $F(t) = t^2 \sin t^{-2}$ if $t \ne 0$ and $F(0) = 0$, then the solution of $x' = F'(x)x$ is $x(t) = e^{F(t)}$, which is ACG_* on $[0, 1]$. Note that $F(t)$ is not a bounded variation on $[0, 1]$; thus, $F(t)$ and $e^F(t)$ are not absolutely continuous on $[0, 1]$.

THEOREM 2.5 [8]. *A function $f\colon [a, b] \to R^n$ is Henstock–Kurzweil integrable on $[a, b]$ if and only if there exists a continuous function F which is ACG_* on $[a, b]$ such that $F'(t) = f(t)$ almost everywhere.*

THEOREM 2.6 [7, 8] (Controlled Convergence Theorem). *If a sequence of Henstock–Kurzweil integrable function $\{f_n\}$ satisfies the following conditions:*

(i) $f_n(x) \to f(x)$ *almost everywhere in $[a, b]$ as $n \to \infty$;*

(ii) *the primitives $F_n(x) = \int_a^b f_n(s)\,ds$ of f_n are ACG_* uniformly in n;*

(iii) *the primitives F_n are equicontinuous on $[a, b]$, then f is Henstock–Kurzweil integrable on $[a, b]$ and we have $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$ as $n \to \infty$.*

If conditions (ii) *and* (iii) *are replaced by the condition*:

(iv) $g(x) \leq f_n(x) \leq h(x)$ *almost everywhere on* $[a, b]$, *where* g *and* h *are Henstock–Kurzweil integrable, then the result of Theorem* 2.6 *also holds*.

3. GENERALIZED CARATHEODORY SYSTEMS

In this section, a generalized Caratheodory system of the form (1.1) is defined by using the Henstock–Kurzweil integral. The main result of this section is an existence theorem for the solution to the generalized Caratheodory system.

DEFINITION 3.1 [2]. Let right-hand side function $f(t, x)$ of the system (1.1) be a Caratheodory function defined on an open region G; i.e., f is continuous in x for almost all t and measurable in t for each fixed x. Suppose there exists a Lebesgue integrable function $m(t)$ on every bounded closed subregion $G_0 \subset G$ such that

$$\| f(t, x) \| \leq m(t), \quad \text{for } (t, x) \in G_0, \tag{3.1}$$

then $f(t, x)$ is said to satisfy the Caratheodory condition, and the system (1.1) is said to be a Caratheodory system.

DEFINITION 3.2 [2]. A function $x(t)$: $I \to R^n$ (I represents an interval in R^1) is said to be a solution of the Caratheodory system (1.1) or, in short, C-solution, if $x(t)$ satisfies the following conditions:

(i) $x(t)$ is absolutely continuous on each compact subinterval of interval I;

(ii) $(t, x) \in G$ for $t \in I$;

(iii) $x'(t) = f(t, x(t))$ for almost everywhere $t \in I$.

Now we can generalize the Caratheodory system to the generalized Caratheodory system using the Henstock–Kurzweil integral.

DEFINITION 3.3. Let the right-hand side function of the system (1.1) be a Caratheodory function defined on an open region G. Suppose that there exist two Henstock–Kurzweil integrable functions $g(t)$ and $h(t)$ for every bounded closed subregion $G_0 \subset G$ such that

$$g(t) \leq f(t, x) \leq h(t), \quad \text{for all } x \text{ and almost all } t \text{ with } (t, x) \in G_0, \tag{3.2}$$

then the $f(t, x)$ satisfies the generalized Caratheodory condition on G, and the system (1.1) is a generalized Caratheodory system.

DEFINITION 3.4. A function $x(t)$: $I \to R^n$ (I represents an interval in R^1) is said to be a solution of the generalized Caratheodory system (1.1) or GC-solution if $x(t)$ satisfies the following conditions:

(i) $x(t)$ is ACG_* on each compact subinterval of I;

(ii) $(t, x) \in G$ for $t \in I$;

(iii) $x'(t) = f(t, x(t))$ for almost everywhere $t \in I$.

We remark that if g and h are Lebesgue integrable functions, then the previous generalized Caratheodory system reduces to the Caratheodory system, and the GC-solution also reduces to the C-solution.

Now we give the existence theorem for the GC-solution for the preceding generalized Caratheodory system.

THEOREM 3.5. *Suppose that $f(t, x)$ satisfies the conditions of Definitions* 3.3, *then there exists a GC-solution Φ of the generalized Caratheodory system* (1.1) *on some interval $|t - \tau| \le \beta$, and satisfies $\Phi(\tau) = \xi$.*

Proof. Let $(\tau, \xi) \in G$ be fixed and let G_0 be a bounded closed subregion of G,

$$G_0: \qquad |t - \tau| \le a, \quad \|x - \xi\| \le b, \tag{3.3}$$

then there exist two Henstock–Kurzweil integrable functions $g(t)$ and $h(t)$ such that for all x and for almost all t with $(t, x) \in G_0$ we have $0 \le f(t, x) - g(t) \le h(t)) - g(t)$. By Theorem 2.3, $h - g$ is Lebesgue integrable. Let

$$F(t, x) = f\left(t, x + \int_\tau^t g(s)\, ds\right) - g(t), \tag{3.4}$$

then F is a Caratheodory function, $0 \le F(t, x) \le h(t) - g(t)$ for all $(t, x) \in G_0'$, where G_0' is a bounded closed subregion of G_0, such that

$$\left\| x + \int_\tau^t g(s)\, ds - \xi \right\| \le b, \quad \text{for all } (t, x) \in G_0',$$

then $x' = F(t, x)$ is a Caratheodory system. By Caratheodory existence theorem, there is a function Ψ on some interval $|t - \tau| \le \beta$ such that $\Psi'(t) = F(t, \Psi(t))$ almost everywhere in this interval and $\Psi(\tau) = \xi$. Let

$$\Phi(t) = \Psi(t) + \int_\tau^t g(s)\, ds. \tag{3.5}$$

Then, for almost all t,

$$\begin{aligned}\Phi'(t) &= \Psi'(t) + g(t)\\ &= F(t, \Psi(t)) + g(t)\\ &= f\left(t, \Psi(t) + \int_\tau^t g(s)\, ds\right) - g(t) + g(t)\\ &= f(t, \Phi(t)),\end{aligned}$$

and

$$\Phi(\tau) = \Psi(\tau) + \int_\tau^\tau g(s)\, ds = \xi.$$

The proof is complete.

EXAMPLE 3. Consider

$$x' = f(t, x) = g(t, x) + h(t), \tag{3.6}$$

where $\|g(t, x)\| \le g_1(t)$ for all $|t| \le 1$, $\|x\| \le 1$, $g_1(t)$ is Lebesgue integrable on $|t| \le 1$, and $h(t) = (d/dt)(t^2 \sin t^{-2})$ if $t \ne 0$ and $h(0) = 0$. Here $\tau = \xi = 0$. h is Henstock–Kurzweil integrable but not Lebesgue integrable and

$$h(t) - g_1(t) \le f(t, x) \le h(t) + g_1(t), \quad \text{for } |t| \le 1, \|x\| \le 1.$$

Thus, by Theorem 3.5, there exists a GC-solution Φ of $x' = f(t, x)$ with $\Phi(0) = 0$. For instance, if $g(t, x) = t^2 x$, then this example is Example 1,

$$\Phi(t) = e^{t^3/3} \int_0^t e^{-s^3/3} h(s)\, ds.$$

THEOREM 3.6. *Let the right-hand side function $f(t, x)$ of the system* (1.1) *be a Caratheodory function on the open region G, Let G_0: $|t - \tau| \le a$, $\|x - \xi\| \le b$, be a fixed bounded closed subregion of G, and let $f(t, w(t))$ be Henstock–Kurzweil integrable on $|t - \tau| \le a$ for any step function $w(t)$ defined on $|t - \tau| \le a$ with values in $\|x - \xi\| \le b$. Denote $F_w(t) = \int_\tau^t f(s, w(s))\, ds$.*

If $\{F_w$: w is a step function$\}$ is ACG_* uniformly in w and equicontinuous on $|t - \tau| \le a$, then the system (1.1) *is a generalized Caratheodory system.*

Proof. Notice that $f(t, x)$ is a Caratheodory function. Thus, there exist two measurable functions $u(t)$ and $v(t)$ defined on $|t - \tau| \le a$ with values in $\|x - \xi\| \le b$ such that

$$f(t, u(t)) \le f(t, x) \le f(t, v(t)), \tag{3.7}$$

for all $(t, x) \in G_0$ [see 14, Lemma 17.2]. Next, we should show that $f(t, u(t))$ and $f(t, v(t))$ are Henstock–Kurzweil integrable by using the controlled convergence Theorem 2.6. First, there is a sequence $\{k_n(t)\}$ of step functions defined on $|t - \tau| \leq a$ with values in $\|x - \xi\| \leq b$ such that $k_n(t) \to u(t)$ almost everywhere as $n \to \infty$. Thus, $f(t, k_n(t)) \to f(t, u(t))$ almost everywhere as $\eta \to \infty$. Let

$$F_n(t) = \int_\tau^t f(s, k_n(s))\, ds.$$

Then $\{F_n(t)\}$ is ACG_* uniformly in n and equicontinuous. By Theorem 2.6, $f(t, u(t))$ is Henstock–Kurzweil integrable. Similarly, $f(t, v(t))$ is Henstock–Kurzweil integrable. Let $g(t) = f(t, u(t))$, $h(t) = f(t, v(t))$, then by (3.7) we have

$$g(t) \leq f(t, x) \leq h(t).$$

The proof is complete.

4. UNIQUENESS THEOREM FOR THE GENERALIZED CARATHEODORY SYSTEM

In this section, the uniqueness theorems for the solutions of the generalized Caratheodory system is given by using some inequalities from the Henstock–Kurzweil integral.

LEMMA 4.1 [8]. *If $f(t)$: $[a, b] \to R^n$ is a Henstock–Kurzweil integrable function with respect to the function $g(t)$: $[a, b] \to R^n$, then, for every $c \in [a, b]$, we have*

$$\lim_{s \to c} \left[\int_a^s f(t)\, dg(t) - f(c)(g(s) - g(c)) \right] = \int_a^c f(t)\, dg(t), \quad (4.1)$$

$$\lim_{u \to c} \left[\int_u^b f(t)\, dg(t) + f(c)(g(u) - g(c)) \right] = \int_c^b f(t)\, dg(t). \quad (4.2)$$

LEMMA 4.2. *Let f, g: $[a, b] \to R^n$ be functions for which the Henstock–Kurzweil-integral $\int_a^b f(t)\, dg(t)$ exists. If u, v: $[a, b] \to R^1$ are functions such that integral $\int_a^b u(t)\, dv(t)$ exists and if there is a positive function δ: $[a, b] \to (0, +\infty)$ such that*

$$|t - \xi|\, \|f(\xi)(g(t) - g(\xi))\| \leq (t - \xi) u(\xi)(v(t) - v(\xi)), \quad (4.3)$$

for every $t \in [\xi - \delta(\xi), \xi + \delta(\xi)]$, $\xi \in [a, b]$. *Then*

$$\left\| \int_a^b f(t)\, dg(t) \right\| \le \int_a^b u(t)\, dv(t). \tag{4.4}$$

Proof. Assume that $\epsilon > 0$ is given. Because the integrals $\int_a^b f(t)\, dg(t)$, $\int_a^b u(t)\, dv(t)$ exist, there is a positive function δ_1 on $[a, b]$ with $\delta_1(s) \le \delta(s)$ for $s \in [a, b]$ such that for every δ_1-fine partition D of $[a, b]$,

$$a = t_0 < t_1 < \cdots < t_k = b \quad \text{and} \quad \{\xi_1, \xi_2, \dots, \xi_k\},$$

with $\xi_i - \delta_1(\xi_i) < t_{i-1} \le \xi_i \le t_i < \xi_i + \delta_1(\xi_i)$, $i = 1, 2, \dots, k$, we have

$$\left\| \sum_{i=1}^k f(\xi_i)\big(g(t_i) - g(t_{i-1})\big) - \int_a^b f(t)\, dg(t) \right\| < \epsilon, \tag{4.5}$$

$$\left| \sum_{i=1}^k u(\xi_i)\big(v(t_i) - v(t_{i-1})\big) - \int_a^b u(t)\, dv(t) \right| < \epsilon. \tag{4.6}$$

By (4.3), for $j = 1, 2, \dots, k$, we have

$$\begin{aligned} \big\| f(\xi_j)\big(g(t_j) - g(t_{j-1})\big) \big\| &\le \big\| f(\xi_j)\big(g(t_j) - g(\xi_j)\big) \big\| \\ &\quad + \big\| f(\xi_j)\big(g(\xi_j) - g(t_{j-1})\big) \big\| \\ &\le u(\xi_j)\big(v(t_j) - v(t_{j-1})\big). \end{aligned}$$

Hence,

$$\begin{aligned} \left\| \int_a^b f(t)\, dg(t) \right\| &\le \left\| \sum_{i=1}^k f(\xi_i)\big(g(t_i) - g(t_{i-1})\big) - \int_a^b f(t)\, dg(t) \right\| \\ &\quad + \left\| \sum_{i=1}^k f(\xi_i)\big(g(t_i) - g(t_{i-1})\big) \right\| \\ &< \epsilon + \sum_{i=1}^k u(\xi_i)\big(v(t_i) - v(t_{i-1})\big) \\ &< 2\epsilon + \int_a^b u(t)\, dv(t). \end{aligned}$$

Because $\epsilon > 0$ is arbitrary, the inequality (4.4) is satisfied.

COROLLARY 4.3. *If* $f\colon [a, b] \to R^1$, $f(t)| \le c$ *for* $t \in [a, b]$ *where* c *is a constant*, $g\colon [a, b] \to R^1$ *is of bounded variation on* $[a, b]$, *and the integral*

$\int_a^b f(t)\,dg(t)$ exists then

$$\left|\int_a^b f(t)\,dg(t)\right| \le c\,\mathrm{Var}_a^b\, g, \tag{4.7}$$

where $\mathrm{Var}_a^b\, g$ *is the total variation of* g *on* $[a,b]$.

LEMMA 4.4. *Let* $h\colon [a,b] \to R^1$ *be a nonnegative nondecreasing function which is continuous from the left on* $(a,b]$. *Assume that* $u\colon [0,+\infty) \to [0,+\infty)$ *is a continuous nondecreasing function with primitive* $U\colon [0,+\infty) \to R^1$, *then the integral* $\int_a^b u(h(t))\,dh(t)$ *exists and*

$$\int_a^b u(h(t))\,dh(t) \le U(h(b)) - U(h(a)). \tag{4.8}$$

Proof. The composition of functions u and h given by $u(h(t))$ for $t \in [a,b]$ is nondecreasing on $[a,b]$. Therefore, the integral $\int_a^b u(h(t))\,dh(t)$ exists. Assume that $\epsilon > 0$ is given. By the definition of the primitive U to u, for every $\in [0,+\infty)$ there exists $\theta(s) > 0$ such that for every $\eta > 0$ with $0 \le |\eta| < \theta(s)$ we have

$$|U(s+\eta) - U(s) - u(s)\eta| \le \epsilon|\eta|. \tag{4.9}$$

Because $\lim_{t\to\xi^+} h(t) = h(\xi^+)$ for $\xi \in [a,b)$, there exists a $\delta^+(\xi) > 0$, $\delta^+(b) = 1$ such that for $t \in (\xi, \xi + \delta^+(\xi)] \cap [a,b]$ we have

$$0 \le h(t) - h(\xi^+) \le \theta(h(\xi^+)).$$

Let $s = h(\xi^+)$ and $\eta = h(t) - h(\xi^+)$ we obtain

$$u(h(\xi^+))(h(t) - h(\xi^+)) \le U(h(t)) - U(h(\xi^+)) + \epsilon(h(t) - h(\xi^+)).$$

Further, we have

$$\begin{aligned} u(h(\xi))&(h(\xi^+) - h(\xi)) - [U(h(\xi^+)) - U(h(\xi))] \\ &= \int_{h(\xi)}^{h(\xi^+)} [u(h(\xi)) - u(s)]\,ds \\ &\ge 0, \end{aligned}$$

because $u(h(\xi)) \le u(s)$ for $s \in [h(\xi), h(\xi^+)]$. Therefore,

$$\begin{aligned} |u(h(\xi))(h(t) - h(\xi))| &= u(h(\xi))(h(t) - h(\xi^+)) \\ &\quad + u(h(\xi))(h(\xi^+) - h(\xi)) \\ &\le U(h(t)) - U(h(\xi)) + \epsilon(h(t) - h(\xi)), \end{aligned}$$

for $t \in (\xi, \xi + \delta^+(\xi)) \cap [a, b]$. From the inequality (4.9) and from the left continuity of the function h at the point $\xi \in (a, b]$, there is a $\delta^-(\xi) > 0$, $\delta^-(a) = 1$ such that for $t \in [\xi - \delta^-(\xi), \xi] \cap [a, b]$ the inequality,

$$|u(h(\xi))(h(t) - h(\xi))| \leq U(h(\xi)) - U(h(t)) + \epsilon(h(\xi) - h(t))$$

is satisfied. Let $\delta(\xi) = \min\{\delta^-(\xi), \delta^+(\xi)\}$ for $\xi \in [a, b]$, then for $\xi \in [a, b]$ and $t \in [\xi - \delta(\xi), \xi + \delta(\xi)] \cap [a, b]$ we obtain by the preceding inequalities the relation,

$$\begin{aligned} &|t - \xi|\,|u(h(\xi))(h(t) - h(\xi))| \\ &\qquad \leq (t - \xi)(U(h(t)) + \epsilon h(t) - U(h(\xi)) - \epsilon h(\xi)). \end{aligned}$$

By Lemma 4.2, this inequality implies

$$\begin{aligned} \int_a^b u(h(t))\, dh(t) &\leq \int_a^b d[U(h(t)) + \epsilon h(t)] \\ &= U(h(b)) - U(h(a)) + \epsilon(h(b) - h(a)). \end{aligned}$$

Because $\epsilon > 0$ can be chosen arbitrarily small, the proof is complete.

LEMMA 4.5. *Let ψ: $[a, b] \to [0, +\infty)$, h: $[a, b] \to [a, +\infty)$ be given where ψ is bounded and h is nondecreasing and continuous from the left on the interval $[a, b]$. Suppose that the function ω: $[0, +\infty) \to R^1$ is continuous, nondecreasing, $\omega(0) = 0$, $\omega(r) > 0$ for $r > 0$. For $u > 0$, let*

$$\Omega(u) = \int_{u_0}^{u} \frac{1}{\omega(r)}\, dr, \tag{4.10}$$

with some $u_0 > 0$.

The function Ω: $(0, +\infty) \to R^1$ is increasing, $\Omega(u_0) = 0$ and $\lim_{u \to 0^-} \Omega(u) = \alpha \geq -\infty$, $\lim_{u \to +\infty} \Omega(u) = \beta \leq +\infty$.

Assume that for $\xi \in [a, b]$ the inequality,

$$\psi(\xi) \leq k + \int_a^{\xi} \omega(\psi(t))\, dh(t) \tag{4.11}$$

holds, where $k > 0$ is a constant.

If $\Omega(k) + h(b) - h(a) < \beta$ then for $\xi \in [a, b]$ we have

$$\psi(\xi) \leq \Omega^{-1}(\Omega(k) + h(\xi) - h(a)), \tag{4.12}$$

where Ω^{-1}: $(\alpha, \beta) \to R^1$ is the inverse function of the function Ω in (4.10).

Proof. If we have $\Omega(l) + h(b) - h(a) < \beta$ for some $l > 0$ then for all $t \in [a, b]$ we have

$$\alpha < \Omega(l) + h(t) - h(a) < \beta.$$

Therefore the value of $\Omega(l) + h(t) - h(a)$ belongs to the domain of Ω^{-1} provided $t \in [a, b]$, and for t we can define

$$H_l(t) = \Omega^{-1}(\Omega(l) + h(t) - h(a)).$$

Define further

$$\varphi(s) = \omega\big(\Omega^{-1}(\Omega(l) + s)\big), \tag{4.13}$$

for $s \in [0, \beta - \Omega(l)]$.

At $\Omega^{-1}(\Omega(l) + s)$ there exists a derivative Ω' of the function Ω and

$$\Omega'\big(\Omega^{-1}(\Omega(l) + s)\big) = \frac{1}{\omega\big(\Omega^{-1}(\Omega(l) + s)\big)} \neq 0.$$

The well-known formula for the derivative of the inverse function leads to

$$\begin{aligned}\frac{d}{ds}\big[\Omega^{-1}(\Omega(l) + s)\big] &= \frac{1}{\Omega'\big(\Omega^{-1}(\Omega(l) + s)\big)} \\ &= \omega\big(\Omega^{-1}(\Omega(l) + s)\big) \\ &= \varphi(s),\end{aligned} \tag{4.14}$$

for $s \in [0, \beta - \Omega(l)]$. If now $\xi \in [a, b]$ is given, then using the definition of the function φ from (4.13) we obtain

$$\begin{aligned}\int_a^\xi \omega(H_l(t))\, dh(t) &= \int_a^\xi \omega\big(\Omega^{-1}(\Omega(l) + h(t) - h(a))\big)\, dh(t) \\ &= \int_a^\xi \varphi(h(t) - h(a))\, d(h(t) - h(a)).\end{aligned}$$

This together with (4.14) and Lemma 4.4 imply

$$\begin{aligned}\int_a^\xi \omega(H_l(t))\, dh(t) &\leq \Omega^{-1}(\Omega(l) + h(\xi) - h(a)) - \Omega^{-1}(\Omega(l)) \\ &= H_l(\xi) - l,\end{aligned}$$

and consequently for $\xi \in [a, b]$ we have the inequality,

$$l + \int_a^{\xi} \omega(H_l(t))\, dh(t) \leq H_l(\xi).$$

Assume that $\epsilon_0 > 0$ is such that $\Omega(k + \epsilon_0) + h(b) - h(a) < \beta$. Let us take an arbitrary $\epsilon \in (0, \epsilon_0)$ and set $l = k + \epsilon$. For this case the last inequality reads

$$k + \epsilon + \int_a^{\xi} \omega(H_{k+\epsilon}(t))\, dh(t) \leq H_{k+\epsilon}(\xi),$$

and taking into account the relation (4.11) for every $\xi \in [a, b]$ we get

$$\begin{aligned} \psi(\xi) - H_{k+\epsilon}(\xi) &\leq k + \int_a^{\xi} \omega(\psi(t))\, dh(t) - k - \epsilon \\ &\qquad - \int_a^{\xi} \omega(H_{k+\epsilon}(t)\, dh(t) \\ &= -\epsilon + \int_a^{\xi} [\omega(\psi(t)) - \omega(H_{k+\epsilon}(t))]\, dh(t). \end{aligned} \tag{4.15}$$

Hence $\psi(a) - H_{k+\epsilon}(\xi) \leq -\epsilon$ and also $\omega(\psi(a)) - \omega(H_{k+\epsilon}(a)) \leq 0$ because the function ω is assumed to be nondecreasing. The functions ψ and $H_{k+\epsilon}$ are bounded and therefore there is a constant $K > 0$ such that

$$|\omega(\psi(t)) - \omega(H_{k+\epsilon}(t))| < K,$$

for $t \in [a, b]$. Using Lemma 4.1 and Corollary 4.3 we obtain from the last two displayed inequalities,

$$\begin{aligned} \psi(\xi) - H_{k+\epsilon}(\xi) &\leq -\epsilon + [\omega(\psi(a)) - \omega(H_{k+\epsilon}(t))](h(a^+) - h(a)) \\ &\qquad + \lim_{\delta \to 0^+} \int_{a+\delta}^{\xi} [\omega(\psi(t)) - \omega(H_{k+\epsilon}(t))]\, dh(t) \\ &\leq -\epsilon + K \lim_{\delta \to 0^+} [h(\xi) - h(a + \delta)] \\ &= -\epsilon + K[h(\xi) - h(a^+)]. \end{aligned}$$

Because $\lim_{\xi \to a^+} h(\xi) = h(a^+)$, an $\eta > 0$ can be found such that for $\xi \in (a, a + \eta)$ the inequality $h(\xi) - h(a^+) < \epsilon/(2K + 1)$ holds and therefore also

$$\psi(\xi) - H_{k+\epsilon}(\xi) < -\epsilon + \frac{K\epsilon}{2K+1} < -\frac{\epsilon}{2} < 0,$$

for $\xi \in (a, a+\eta)$.

Let us set

$$T = \sup\{t \in [a,b]; \psi(\xi) - H_{k+\epsilon}(\xi) < 0, \text{for } \xi \in [a,t]\}.$$

As has been shown previously, we have $T > a$ and for $\xi \in [a,T)$ the inequality $\psi(\xi) - H_{k+\epsilon}(\xi) < 0$ and therefore $\omega(\psi(\xi)) - \omega(H_{k+\epsilon}(\xi)) \leq 0$ holds. The last conclusion is a consequence of the assumption that ω is nondecreasing. By (4.15) and Lemma 4.1 we have

$$\begin{aligned}\psi(T) - H_{k+\epsilon}(T) &\leq -\epsilon + \lim_{\delta\to 0^+}\int_a^{T-\delta}\left[\omega(\psi(t)) - \omega\left(H_{k+\epsilon}(t)\right)\right] dh(t) \\ &\qquad + \left[\omega(\psi(T)) - \omega\left(H_{k+\epsilon}(T)\right)\right]\left(h(T) - h(T^-)\right) \\ &\leq -\epsilon \\ &< 0,\end{aligned}$$

because

$$h(T) - h(T^-) = h(T) - \lim_{t\to T^-} h(t) = 0,$$

and

$$\lim_{\delta\to 0^+}\int_a^{T-\delta}\left[\omega(\psi(t)) - \omega\left(H_{k+\epsilon}(t)\right)\right] dh(t) \leq 0.$$

If we assume that $T < b$ then we can repeat this procedure for $\xi > T$ by virtue of the inequality,

$$\psi(\xi) - H_{k+\epsilon}(\xi) \leq -\epsilon + \int_T^{\xi}\left[\omega(\psi(t)) - \omega\left(H_{k+\epsilon}(t)\right)\right] dh(t),$$

thus obtaining $\psi(\xi) - H_{k+\epsilon}(\xi) < 0$ for $\xi \in [T, T+\eta]$ for some $\eta > 0$. Hence $T = b$ and

$$\psi(\xi) < H_{k+\epsilon}(\xi) = \Omega^{-1}\left(\Omega(k+\epsilon) + h(\xi) - h(a)\right),$$

for $\xi \in [a,b]$. Because the function Ω is continuous and the last inequality holds for every sufficiently small $\epsilon > 0$, we obtain the inequality (4.12). The proof is complete.

COROLLARY 4.6. *If ψ, h, and k satisfy the assumptions in lemma* 4.5 *and if for $\xi \in [a, b]$ the inequality,*

$$\psi(\xi) \le k + L \int_a^{\xi} \psi(t)\, dh(t)$$

holds with a constant $L > 0$ instead of (4.11), *then for every $\xi \in [a, b]$ the inequality,*

$$\psi(\xi) \le k e^{L(h(\xi) - h(a))}$$

is satisfied.

Lemma 4.5 represents a Bellman type inequality for the Henstock–Kurzweil integral. Results of this type are especially useful for deriving uniqueness results for the generalized Caratheodory system.

DEFINITION 4.7. A solution x: $[\tau, \tau + \eta] \to R^n$ of the generalized Caratheodory system (1.1) is called locally unique for increasing values of t if for any solution y: $[\tau, \tau + \sigma] \to R^n$ of the generalized Caratheodory system (1.1) with $y(\tau) = x(\tau)$ there exists $\eta_1 > 0$ such that $x(t) = y(t)$ for $t \in [\tau, \tau + \eta] \cap [\tau, \tau + \sigma] \cap [\tau, \tau + \eta_1]$.

THEOREM 4.8. *Assume that the system* (1.1) *is a generalized Caratheodory system and the right-hand side function $f(t, x)$ of system* (1.1) *satisfies the following condition on G_0: $|t - \tau| \le a$, $\|x - \xi\| \le b$,*

$$\|f(\theta, x) - f(\theta, y)\|(v - u) \le \omega(\|x - y\|)(h(v) - h(u)), \quad (4.16)$$

for each interval $[u, v]$ with $\theta \in [u, v] \subset [\tau - a, \tau + a]$ and all x, y belonging to $\|x - \xi\| \le b$, where h: $[\tau - a, \tau + a] \to R^1$ is nondecreasing and continuous from the left. ω: $[0, +\infty) \to R^1$ is continuous, nondecreasing, $\omega(r) > 0$ for $r > 0$, $\omega(0) = 0$ and

$$\lim_{v \to 0^+} \int_v^u \frac{1}{\omega(r)}\, dr, \quad (4.17)$$

for every $u > 0$. Then every solution x of the generalized Caratheodory system (1.1) *such that $x(\tau) = \xi$ is locally unique for increasing values of t.*

Proof. Let $x, y\colon [\tau, \tau+\eta] \to R^n$ be solutions of the generalized Caratheodory system (1.1) such that $x(\tau) = y(\tau) = \xi$. Assume that $\epsilon > 0$ is given. Because the Henstock–Kurzweil integrals $\int_r^t [f(s, x(s)) - f(s, y(s))]\, ds$, $\int_\tau^t \omega(\|x(s) - y(s)\|)\, dh(s)$ exist there is a positive function δ_1 on $[\tau, t]$ such that for every δ_1-fine partition D of $[\tau, t]$,

$$\tau = t_0 < t_1 < \cdots < t_k = t \quad \text{and} \quad \{\theta_1, \theta_2, \ldots, \theta_k\},$$

with $\theta_i - \delta_1(\theta_i) < t_{i-1} \le \theta_i \le t_i < \theta_i + \delta_1(\theta_i)$, $i = 1, 2, \ldots, k$, we have

$$\begin{aligned}
&\left\| \int_\tau^t [f(s, x(s)) - f(s, y(s))]\, ds \right\| \\
&\quad \le \left\| \int_\tau^t [f(s, x(s)) - f(s, y(s))]\, ds \right. \\
&\qquad \left. - \sum_{i=1}^k [f(\theta_i, x(\theta_i)) - f(\theta_i, y(\theta_i))](t_i - t_{i-1}) \right\| \\
&\qquad + \left\| \sum_{i=1}^k [f(\theta_i, x(\theta_i)) - f(\theta_i, y(\theta_i))](t_i - t_{i-1}) \right\| \\
&\quad < \frac{\epsilon}{2} + \sum_{i=1}^k \omega(\|x(\theta_i) - y(\theta_i)\|)(h(t_i) - h(t_{i-1})),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^k \omega(\|x(\theta_i) - y(\theta_i)\|)(h(t_i) - h(t_{i-1})) \\
&\quad \le \left\| \sum_{i=1}^k \omega(\|x(\theta_i) - y(\theta_i)\|)(h(t_i) - h(t_{i-1})) \right. \\
&\qquad\qquad \left. - \int_\tau^t \omega(\|x(s) - y(s)\|)\, dh(s) \right\| \\
&\qquad + \int_\tau^t \omega(\|x(s) - y(s)\|)\, dh(s) \\
&\quad < \frac{\epsilon}{2} + \int_\tau^t \omega(\|x(s) - y(s)\|)\, dh(s),
\end{aligned}$$

then

$$\|x(t) - y(t)\| = \left\| \int_\tau^t [f(s, x(s)) - f(s, y(s))] \, ds \right\|$$

$$< \epsilon + \int_\tau^t \omega(\|x(s) - y(s)\|) \, dh(s).$$

Because $\epsilon > 0$ is arbitrary, then

$$\|x(t) - y(t)\| \le \int_\tau^t \omega(\|x(s) - y(s)\|) \, dh(s)$$

$$= \int_\tau^{\tau+\delta} \omega(\|x(s) - y(s)\|) \, dh(s)$$

$$+ \int_{\tau+\delta}^t \omega(\|x(s) - y(s)\|) \, dh(s),$$

where $0 < \delta < t - \tau$. By Lemma 4.1, we have

$$\begin{aligned}
&\int_\tau^{\tau+\delta} \omega(\|x(s) - y(s)\|) \, dh(s) \\
&\quad = \omega(\|x(\tau) - y(\tau)\|)[h(\tau^+) - h(\tau)] \\
&\qquad + \lim_{t_1 \to \tau^+} \int_{t_1}^{\tau+\delta} \omega(\|x(s) - y(s)\|) \, dh(s) \\
&\quad \le \sup_{s \in [\tau, \tau+\delta]} \omega(\|x(s) - y(s)\|)[h(\tau + \delta) - h(\tau^+)] \\
&\quad = A(\delta),
\end{aligned} \tag{4.18}$$

because $\omega(\|x(\tau) - y(\tau)\| = \omega(0) = 0$.

Because the limit $h(\tau^+)$ exists we have also $\lim_{\delta \to 0^+} A(\delta) = 0$, therefore,

$$\|x(t) - y(t)\| \le A(\delta) + \int_{\tau+\delta}^t \omega(\|x(s) - y(s)\|) \, dh(s),$$

for $t \in [\tau + \delta, \tau + \eta]$.

Let $u_0 > 0$ and set

$$\Omega(u) = \int_{u_0}^u \frac{1}{\omega(r)} \, dr.$$

Using Lemma 4.5 we obtain

$$\|x(t) - y(t)\| \le \Omega^{-1}(\Omega(A(\delta)) + h(t) - h(\tau+\delta)), \quad (4.19)$$

for $t \in [\tau+\delta, \tau+\eta]$ provided $\Omega(A(\delta)) + h(\tau+\eta) - h(\tau+\delta) < \beta$, where $\beta = \lim_{u\to+\infty} \Omega(u) \le +\infty$. Evidently, we have

$$\Omega(A(\delta)) + h(\tau+\eta) - h(\tau+\delta) \le \Omega(A(\delta)) + h(\tau+\eta) - h(\tau^+),$$

and because $\lim_{\delta\to 0^+} A(\delta) = 0$ and $\lim_{u\to 0^+} \Omega(u) = -\infty$, we have

$$\lim_{\delta\to 0^+} \Omega(A(\delta)) + h(\tau+\eta) - h(\tau^+) = -\infty.$$

Hence there is a $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$ the inequality $\Omega(A(\delta)) + h(\tau+\eta) - h(\tau^+) < \beta$ holds. Applying now the map Ω to both sides of (4.19), we have

$$\Omega(\|x(t) - y(t)\|) \le \Omega(A(\delta)) + h(t) - h(\tau+\eta),$$

and this yields

$$\Omega(\|x(t) - y(t)\|) - \Omega(A(\delta)) \le h(t) - h(\tau+\delta) \le h(t) - h(\tau^+).$$

From the definition of Ω we therefore have

$$\int_{A(\delta)}^{\|x(t)-y(t)\|} \frac{1}{\omega(r)}\, dr \le h(\tau+\eta) - h(\tau^+).$$

Assume now that $\|x(t^*) - y(t')\| = k > 0$ for some $t^* \in (\tau, \tau+\eta]$, then

$$\int_{A(\delta)}^{k} \frac{1}{\omega(r)}\, dr \le h(\tau+\eta) - h(\tau^+) < +\infty,$$

for every $\delta \in (0, \delta_0)$ such that $\delta < t^* - \tau$. Now it is possible to use $\delta \to 0^+$ to obtain the inequality,

$$\lim_{\delta\to 0^+} \int_{A(\delta)}^{k} \frac{1}{\omega(r)}\, dr \le h(\tau+\eta) - h(\tau^+) < +\infty,$$

which contradicts the assumption on the function ω. Therefore $\|x(t) - y(t)\| = 0$ for $t \in (\tau, \tau+\eta]$. The proof is complete.

COROLLARY 4.9. *If the function* $\omega(r) = Lr$, $r > 0$, $L > 0$ *in Theorem* 4.8, *then the result in Theorem* 4.8 *also holds*.

The local uniqueness for increasing values of t can be extended to the global uniqueness for increasing values of t in the same manner as is done for the case of classical ordinary differential equations, we elaborate on this.

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